A SPACE OF MULTIPLIERS ON L

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Abstract. Conditions for a function (number sequence) to be a multiplier on the space of integrable functions on $\mathbb{R}$ ($\mathbb{T}$) are given. This generalizes recent results of Giang and Moricz.

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1. Introduction

In their recent paper [GM], Dăng Vũ Giang and F. Móricz gave a family of spaces, so that each element of such space is a multiplier on

$$L = \{f : ||f||_L = \int\limits_{\mathbb{R}} |f(x)|\, dx < \infty\},$$

where $\mathbb{R} = (-\infty, \infty)$. These results are obtained both in periodic and non-periodic cases. Proofs are strongly based on some sufficient conditions for a function to have an integrable Fourier transform, or for a trigonometric series to be a Fourier series of an integrable function, respectively.

More general conditions of such type were given in our recent paper [L]. Thus we can give less restrictive multiplier conditions, that is the results of [GM] follow from ours immediately. Some of our notation is the same as in [GM]. It allows us to compare easily our results.

Let f be an integrable function on $\mathbb{R}$, and

$$\hat{f}(x) = \int\limits_{\mathbb{R}} f(t) e^{-ixt}\, dt$$

be its Fourier transform.

We say that a measurable bounded function λ is an M-multiplier if for every $f \in L$ there exists a function $g \in L$ such that

$$\lambda(t)\hat{f}(t) = \hat{g}(t). \tag{1}$$

The norm of the corresponding operator $\Lambda : L \to L$ which assigns to each $f \in L$ the function $\Lambda f = g$ accordingly to (1) may be calculated as usually:

$$||\Lambda||_M = \sup_{||f||_L \le 1} \frac{||\Lambda f||_L}{||f||_L}.$$

One may consider the space $\mathcal{B}$ of absolutely continuous functions on $\mathbb{R}$, bounded over $\mathbb{R}$, and endowed with the norm

$$||\lambda||_{\mathcal{B}} = ||\lambda||_B + \int\limits_{\mathbb{R}} |\lambda'(t)|\, dt,$$

where $||\lambda||_B = \sup_{t\in\mathbb{R}} |\lambda(t)|$. There exist functions in $\mathcal{B}$ which are not multipliers (see, e.g., [T], p.170-172). Thus it is interesting to study some subspaces of $\mathcal{B}$ which are the spaces of multipliers.

2. Description of the space of multipliers

We introduce a subspace of $\mathcal{B}$ denoted by $\mathcal{H}$ and defined as follows. Let

$$S_f = \int_0^\infty \left| \int\limits_{|t|\le \frac{u}{2}} \frac{f(u-t) - f(u+t)}{t}\, dt \right| du.$$

Then, denoting

$$\mathcal{A} = \int\limits_0^\infty \frac{|\lambda(t) - \lambda(-t)|}{t}\, dt,$$

we set

$$\mathcal{H} = \{\lambda : ||\lambda||_{\mathcal{H}} = ||\lambda||_{\mathcal{B}} + S_{\lambda'} + \mathcal{A} < \infty\}.$$

Our main result is the following.

Theorem 1. *If $\lambda \in \mathcal{H}$ then λ is an M- multiplier, and*

$$||\Lambda||_M \leq C||\lambda||_{\mathcal{H}}. \tag{2}$$

Here and in what follows C will mean absolute constants, and C with indices, say C_p, will denote some constants depending only on the indices mentioned. The same letter may denote constants different in different places.

Let us make some remarks on the space $\mathcal{H}$ and compare Theorem 1 with earlier results.

Let f_+ be the odd continuation of the part of a function f supported on $[0, \infty)$, and f_- be the odd continuation of the part of f supported on $(-\infty, 0]$.

Let, further, ReH be the Hardy space with the norm

$$||f||_H = ||f||_L + ||\tilde{f}||_L < \infty,$$

where

$$\tilde{f}(x) = \frac{1}{\pi} \int\limits_{\mathbb{R}} \frac{f(t)}{x-t}\, dt$$

is the Hilbert transform of f.

We say that $f \in H_s$ if $f_+ \in ReH$ and $f_- \in ReH$. It was proved in [L] that $||f||_1 + S_f < \infty$ is equivalent to the fact that $f \in H_s$.

In [GM], the main theorem is similar to our Theorem 1, but with one of the spaces of a family $\{\mathcal{B}_p : 1 \leq p < \infty\}$ instead of $\mathcal{H}$. This family was defined as follows. For $1 < p < \infty$ set

$$\mathcal{A}_q f = \int\limits_0^\infty \left(\frac{1}{u} \int\limits_{u \leq |t| \leq 2u} |f(t)|^q\, dt \right)^{\frac{1}{q}} du$$

where $\frac{1}{p} + \frac{1}{q} = 1$ here and in what follows, while for $p = 1$

$$\mathcal{A}_\infty f = \int\limits_0^\infty \operatorname*{ess\,sup}_{u \leq |t| \leq 2u} |f(t)|\, du.$$

Then for $1 \leq p < \infty$

$$\mathcal{B}_p = \{\lambda : ||\lambda||_{\mathcal{B}_p} = ||\lambda||_B + \mathcal{A}_q \lambda' + \mathcal{A} < \infty\}.$$

A theorem claimed as the main one in [GM] may be formulated like Theorem 1, with $\mathcal{B}_p$ instead of $\mathcal{H}$. Its proof, after some simple computations, follows from the following

Lemma. (See Lemma 1 in [GM].) *If a function λ is locally absolutely continuous, satisfies the condition*

$$\lim \lambda(t) = 0, \tag{3}$$

and for some $1 < q \le \infty$ *we have* $\mathcal{A}_q < \infty$, *then* $\hat{\lambda}$ *belongs to* L *if and only if* $\mathcal{A} < \infty$. *Furthermore,*

$$||\hat{\lambda}||_L \le \mathcal{A} + C_q \mathcal{A}_q.$$

Indeed, the fact that $\lambda' \in L^1$ yields easily that λ has finite limits l_+ and l_- at $+\infty$ and $-\infty$, respectively. When they coincide, $l_+ = l_- = l$ (and this follows from $\mathcal{A} < \infty$), one can take

$$\lambda_0(t) = \lambda(t) - l$$

and

$$\lambda_1(t) = \hat{\lambda}_0(-t).$$

If $\lambda_1 \in L$ then $\hat{\lambda}_1 = \lambda_0$, and taking

$$\Lambda f = lf + \lambda_1 * f$$

one gets

$$\begin{aligned}(\Lambda f)(t) &= l\hat{f}(t) + \hat{\lambda}_1 \hat{f}(t)\\ &= l\hat{f}(t) + \lambda_0(t)\hat{f}(t) = \lambda(t)\hat{f}(t),\end{aligned}$$

and λ is a multiplier on L.

Theorem 1 may be proved analogously, with application of one our result from [L] instead of Lemma.

Theorem A. (See [L], Theorem 2.) *Let* λ *be a locally absolutely continuous function, satisfying* (3). *Then for* $|x| > 0$ *we have*

$$\hat{\lambda}(x) = \frac{i}{x}\left(\lambda(\frac{\pi}{2|x|}) - \lambda(-\frac{\pi}{2|x|})\right) + \theta\gamma(x),$$

where $|\theta| \le C$, *and*

$$\int\limits_{\mathbb{R}} |\gamma(x)|\, dx \le ||\lambda'||_L + S_{\lambda'}.$$

It is obvious that, in conditions of Theorem A, $\hat{\lambda} \in L$ iff $\mathcal{A} < \infty$. As it was said above, in order to prove Theorem 1 it remains to repeat the proof of Theorem 1 from [GM] using Theorem A instead of Lemma 1.

Indeed, the following embeddings are almost obvious:

$$\mathcal{B}_1 \subset \mathcal{B}_{p_1} \subset \mathcal{B}_{p_2} \subset \mathcal{B}, \qquad 1 < p_1 < p_2 < \infty,$$

while the following fact, proved in [L], is not so clear:

$$\mathcal{B}_p \subset \mathcal{H}, \qquad 1 \le p < \infty.$$

Therefore, the main result in [GM] is contained in Theorem 1.

That $\lambda \in \mathcal{A}_q$ may not be in ReH can be seen from the following counterexample. Let $\lambda(x) = \frac{1}{1+x^2}$. We have $\mathcal{A}_q\lambda < \infty$. Nevertheless $\tilde{\lambda}(x) = \frac{x}{1+x^2}$ (see, e.g., [BN], p.510), and $\tilde{\lambda} \notin L$.

Easy sufficient condition for an even function λ defined on $[0, \infty)$ to be a Fourier multiplier, due to [BN], p.248, has the following relation (so-called quasiconvexity)

$$\int_0^\infty t|d\lambda'(t)| < \infty$$

as the main part. It is easy to verify that this condition is more restrictive than $S_{\lambda'} < \infty$. Indeed, we have

$$S_{\lambda'} = \int_0^\infty \left| \int_0^{\frac{u}{2}} \frac{dx}{x} \int_{u-x}^{u+x} d\lambda'(t) \right| du$$

$$\leq \int_0^\infty du \int_{\frac{u}{2}}^{\frac{3u}{2}} |d\lambda'(t)| \ln \frac{u}{2|u-t|} = \ln 3 \int_0^\infty t|d\lambda'(t)|.$$

3. The case of Fourier series

Analogous results for the case of Fourier series were obtained in [GM] as well. We can generalize these results in the same manner as in the case of Fourier transforms.

Let now L be the space of all complex-valued 2π-periodic functions integrable over $\mathbb{T} = (-\pi, \pi]$, and

$$||f||_L = \int_{\mathbb{T}} |f(x)|\, dx.$$

Let

$$\hat{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx}\, dx, \qquad k = 0, \pm 1, \pm 2, ...$$

be the Fourier coefficients of the function f.

A bounded sequence $\{\lambda = \lambda(k)\}$, with $||\lambda||_m = \sup |\lambda(k)| < \infty$, is called an M-multiplier if for every $f \in L$ there exists a function $g \in L$ such that

$$\lambda(k)\hat{f}(k) = \hat{g}(k), \qquad k = 0, \pm 1, \pm 2, ... \tag{4}$$

As above (4) assigns a bounded linear operator Λ, and it is worth studying spaces of multipliers which are subspaces of the space

$$bv = \{\lambda : ||\lambda||_{bv} = ||\lambda||_m + ||\Delta\lambda||_1 < \infty\},$$

where $|| \cdot ||_1$ is the norm in l^1, and

$$\Delta\lambda(k) = \begin{cases} \lambda(k) - \lambda(k+1) & \text{if} \quad k \geq 0, \\ \lambda(k) - \lambda(k-1) & \text{if} \quad k < 0. \end{cases}$$

Again a sequence $\lambda \in bv$ exists which is not a multiplier (see [Z], Vol.1, p.184).

We introduce a subspace of bv

$$b = \{\lambda : ||\lambda|| = ||\lambda|| + \dots + \dots < \infty\}$$

where

$$s_\lambda = \sum_{m=2}^{\infty} \left| \sum_{k=1}^{[\frac{m}{2}]} \frac{\Delta\lambda(m-k) - \Delta\lambda(m+k)}{k} \right|$$

and

$$a = \sum_{k=1}^{\infty} \frac{|\lambda(k) - \lambda(-k)|}{k}.$$

Note that the condition $s_\lambda < \infty$ is called the Boas-Telyakovskii condition (see, e.g., [T1]).

The analog of Theorem 1 for Fourier series may be formulated as follows.

Theorem 2. *If $\lambda \in h$ then λ is an M-multiplier, and*

$$||\Lambda||_M \le C||\lambda||_h.$$

The proof again may be reduced to the proof of the multiplier theorem for Fourier series in [GM] with application of the following corollary to Theorem A instead of corresponding weaker result in [GM] (see Lemma 3).

Let $\ell(x) = \lambda(k) + (k-x)\Delta\lambda(k)$ for $x \in [k-1,k]$, with $\lim\limits_{|k|\to\infty} \lambda(k) = 0$.

Theorem B. (see [L], Theorem 5). *For every y, $0 < |y| \le \pi$,*

$$\sum_{k=-\infty}^{\infty} \lambda(k)e^{iky} = \frac{i}{y}\left(\ell(\frac{\pi}{2|y|}) - \ell(-\frac{\pi}{2|y|})\right) + \theta\gamma(y) \tag{4}$$

where $\theta \le C$, and

$$\int\limits_{\mathbb{T}} |\gamma(y)|\,dy \le ||\Delta\lambda||_1 + s_\lambda.$$

This is a somewhat stronger form of Telyakovskii's result in [T1].

It is obvious now that the function ℓ, having the sequence λ as its Fourier coefficients, is integrable over $\mathbb{T}$ when $\lambda \in h$. Thus it is enough to substitute this result for Lemma 3 in [GM], and so changed proof establishes Theorem 2.

Analogously to the case of Fourier transforms, the multiplier properties in the case of Fourier series were proved in [GM] for a family of sequences $\{bv_p, 1 \le p < \infty\}$. Each family is a subspace of bv and is defined as follows. Let $I_n = \{2^n, 2^n+1, ..., 2^{n+1}-1\}$, and

$$a_q = \sum_{n=0}^{\infty} 2^n \left(2^{-n} \sum_{|k|\in I_n} |\Delta\lambda(k)|^q\right)^{\frac{1}{q}}$$

for $1 < p < \infty$, while for $p = 1$

$$a_\infty = \sum_{n=0}^{\infty} \max_{|k|\in I_n} |\Delta\lambda(k)|.$$

Then for $1 \le p < \infty$

$$bv_p = \{\lambda : ||\lambda|| \ldots = ||\lambda|| \ldots + a_q + a < \infty\}$$

It is well known that

$$bv_1 \subset bv_{p_1} \subset bv_{p_2} \subset bv, \qquad 1 < p_1 < p_2 < \infty$$

but for us more important is that for all p

$$bv_p \subset h.$$

This was proved for $p = 1$ by Telyakovskii [T2], and for $p > 1$ by Fomin [F]. Therefore the result for Fourier series in [GM] is contained in Theorem 2 as the partial case.

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